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Calculating effective resistances on underlying networks of association schemes

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Abstract

Recently, in Refs. [1] and [2], calculation of effective resistances on distance-regular networks was investigated, where in the first paper, the calculation was based on stratification and Stieltjes function associated with the network, whereas in the latter one a recursive formula for effective resistances was given based on the Christoffel-Darboux identity. In this paper, evaluation of effective resistances on more general networks which are underlying networks of association schemes is considered, where by using the algebraic combinatoric structures of association schemes such as stratification and Bose-Mesner algebras, an explicit formula for effective resistances on these networks is given in terms of the parameters of corresponding association schemes. Moreover, we show that for particular underlying networks of association schemes with diameter d such that the adjacency matrix A possesses $d+1$ distinct eigenvalues, all of the other adjacency matrices A_i, $i \neq 0, 1$ can be written as polynomials of A, i.e., $A_i = P_i(A)$, where P_i is not necessarily of degree i. Then, we use this property for these particular networks and assume that all of the conductances except for one of them, say $c \equiv c_1 = 1$, are zero to give a procedure for evaluating effective resistances on these networks. The preference of this procedure is that one can evaluate effective resistances by using the structure of their Bose-Mesner algebra without any need to know the spectrum of the adjacency matrices.

Keywords: Association scheme, Resistor networks, Stratification, effective resistance, Spectral distribution

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1 Introduction

A classic problem in electric circuit theory studied by numerous authors over many years, is the computation of the resistance between two nodes in a resistor network (see, e.g., [4]). The effective resistance has a probabilistic interpretation based on classical random walker walking on the network. Indeed, the connection between random walks and electric networks has been recognized for some time (see e.g. [5, 6, 7]), where one can establish a connection between the electrical concepts of current and voltage and corresponding descriptive quantities of random walks regarded as finite state Markov chains (for more details see [8]). Also, by adapting the random-walk dynamics and mean-field theory it has been studied that [9], how the growth of a conducting network, such as electrical or electronic circuits, interferes with the current flow through the underlying evolving graphs. In [10], the authors have been shown that, there is also connection between the mathematical techniques for investigating CTQW on graphs, such as Hilbert space of the walk based on stratification and spectral analysis, and electrical concept of resistance between two arbitrary nodes of regular networks and the same techniques can be employed for calculating the resistance. Recently, in Refs. [1] and [2], calculation of effective resistances on distance-regular networks was investigated, where in the first paper, the calculation was based on stratification and Stieltjes function associated with the network, whereas in the latter one a recursive formula for effective resistances was given based on the Christoffel-Darboux identity for orthogonal polynomials. In this paper, we consider more general resistor networks which are underlying networks of association schemes. In fact, the theory of association schemes [11] (the term of association scheme was first coined by R. C. Bose and T. Shimamoto in [12]) has its origin in the design of statistical experiments. The connection of association schemes to algebraic codes, strongly regular graphs, distance-regular graphs, design theory etc., further intensified their study. A further step in the study of association schemes was their algebraization. This formulation was done by R. C. Bose

and D. M. Mesner who introduced an algebra generated by the adjacency matrices of the association scheme, known as Bose-Mesner algebra. We will employ the algebraic structures of the underlying networks of association schemes in order to calculate the effective resistances between arbitrary nodes of them in terms of the parameters of the corresponding association scheme such as diameter of the scheme, the so-called first eigenvalue matrix P, the valencies of the adjacency matrices and the rank of the corresponding idempotents. As we will see, the preference of this employment is that we able to give analytical formulas for effective resistances on these networks in terms of the known parameters of the corresponding association schemes. As it will be shown in section 5, in order to calculate the effective resistances on underlying networks of association schemes, one needs to know the spectrum of the adjacency matrices A_i for $i = 1, ..., d$. Although, in the most cases the spectrum of the Bose-Mesner algebra is known (for example in the cases of group association schemes), but the formulas for effective resistances in terms of the spectrum of the networks do not possess a closed form and evaluation of them in the most cases is not an easy task. So, first we assume that all of the conductances except for one of them, say $c \equiv c_1 = 1$, are zero and consider particular underlying networks of association schemes such that the adjacency matrices A_i can be written as polynomials of the first adjacency matrix $A = A_1$ (not necessarily of degree i). Then, we give a procedure for evaluating the effective resistances on these networks such that one can calculate the effective resistances by using the structure of their Bose-Mesner algebra without any need to know the spectrum of the adjacency matrices.

The organization of the paper is as follows: In section 2, we review some definitions and properties related to association schemes, underlying resistor networks of them and corresponding stratifications. In section 3, the effective resistance in general resistor networks and underlying resistor networks of association schemes is reviewed. Section 4 is devoted to calculation of the effective resistances on underlying resistor networks of association schemes by using the algebraic combinatoric structures of corresponding association schemes without using

the spectrum of underlying networks. In section 5, explicit formula for effective resistances on underlying resistor networks of association schemes is given in terms of spectrum of underlying networks. The paper is ended with a brief conclusion and an appendix.

2 Underlying resistor networks of association schemes

In this section, we review some preliminary tools about underlying networks which are considered through this paper. For material not covered in this section, as well as more detailed information about association schemes and their underlying graphs, refer to [11], [12] and [3].

Definition 1 Assume that V and E are vertex and edge sets of a regular resistor network, respectively (each edge has a certain conductance). Then, the relations $\{R_i\}_{0\leq i\leq d}$ on $V\times V$ satisfying the following conditions

(1) $\{R_i\}_{0\leq i\leq d}$ is a partition of $V\times V$

(2) $R_0=\{(\alpha,\alpha):\alpha\in V\}$

(3) $R_i=R_i^t$ for $0\leq i\leq d$, where $R_i^t=\{(\beta,\alpha):(\alpha,\beta)\in R_i\}$

(4) For $(\alpha,\beta)\in R_k$, the number $p^k_{i,j}=\mid\{\gamma\in X:(\alpha,\gamma)\in R_i \;\; and \;\; (\gamma,\beta)\in R_j\}\mid$ does not depend on (α,β) but only on i,j and k,

define a symmetric association scheme of class d on V which is denoted by $Y=(V,\{R_i\}_{0\leq i\leq d})$. Furthermore, if we have $p^k_{ij}=p^k_{ji}$ for all $i,j,k=0,2,...,d$, then Y is called commutative.

Let $Y=(V,\{R_i\}_{0\leq i\leq d})$ be a commutative symmetric association scheme of class d, then the matrices $A_0,A_1,...,A_d$ defined by

$$\big(A_i\big)_{\alpha,\beta}=\begin{cases}1 & \text{if } (\alpha,\beta)\in R_i,\\ 0 & \text{otherwise} \qquad\qquad (\alpha,\beta\in V)\end{cases} \tag{2-1}$$

are adjacency matrices of Y such that

$$A_iA_j=\sum_{k=0}^{d}p^k_{ij}A_k. \tag{2-2}$$

From (2-2), it is seen that the adjacency matrices $A_0, A_1, ..., A_d$ form a basis for a commutative algebra A known as the Bose-Mesner algebra of Y. This algebra has a second basis $E_0, ..., E_d$ such that

$$E_0 = \frac{1}{N}J, \qquad E_iE_j = \delta_{ij}E_i, \qquad \sum_{i=0}^{d} E_i = I. \tag{2-3}$$

where, $N := |V|$ and J is an $N \times N$ all-one matrix in A. The basis E_i, for $0 \le i \le d$ are known as primitive idempotents of Y. Let P and Q be the matrices relating the two bases for A:

$$A_j = \sum_{i=0}^{d} P_{ij}E_i, \qquad 0 \le j \le d,$$

$$E_j = \frac{1}{N}\sum_{i=0}^{d} Q_{ij}A_i, \qquad 0 \le j \le d. \tag{2-4}$$

Then clearly

$$PQ = QP = NI. \tag{2-5}$$

It also follows that

$$A_jE_i = P_{ij}E_i, \tag{2-6}$$

which shows that the P_{ij} (resp. Q_{ij}) is the i-th eigenvalue (resp. the i-th dual eigenvalue) of A_j (resp. E_j) and that the columns of E_i are the corresponding eigenvectors. Thus $m_i =$ rank(E_i) is the multiplicity of the eigenvalue P_{ij} of A_j (provided that $P_{ij} \neq P_{kj}$ for $k \neq i$). We see that $m_0 = 1, \sum_i m_i = N$, and $m_i =$trace$E_i = N(E_i)_{jj}$ (indeed, E_i has only eigenvalues 0 and 1, so rank(E_k) equals to the sum of the eigenvalues).

Clearly, each non-diagonal (symmetric) relation R_i of an association scheme $Y = (V, \{R_i\}_{0\le i\le d})$ can be thought of as the network (V, R_i) on V, where we will call it the underlying network of association scheme Y. In other words, the underlying network $\Gamma = (V, R_1)$ of an association scheme is an undirected connected network, where the set V and R_1 consist of its vertices and edges, respectively. Obviously replacing R_1 with one of the other relations such as R_i, for $i \neq 0, 1$ will also give us an underlying network $\Gamma = (V, R_i)$ (not necessarily a connected network) with the same set of vertices but a new set of edges R_i.

An undirected connected network $\Gamma = (V, R_1)$ is called distance-regular network if the relations are based on distance function defined as follows: Let the distance between $\alpha, \beta \in V$ denoted by $\partial(\alpha, \beta)$ is the length of the shortest walk connecting α and β (recall that a finite sequence $\alpha_0, \alpha_1, ..., \alpha_n \in V$ is called a walk of length n if $\alpha_{k-1} \sim \alpha_k$ for all $k = 1, 2, ..., n$, where $\alpha_{k-1} \sim \alpha_k$ means that α_{k-1} is adjacent with α_k), then the relations R_i in distance-regular networks are defined as: $(\alpha, \beta) \in R_i$ if and only if $\partial(\alpha, \beta) = i$, for $i = 0, 1, ..., d$, where $d :=\max\{\partial(\alpha, \beta) : \alpha, \beta \in V\}$ is called the diameter of the network. Since $\partial(\alpha, \beta)$ gives the distance between vertices α and β, ∂ is called the distance function. Clearly, we have $\partial(\alpha, \alpha) = 0$ for all $\alpha \in V$ and $\partial(\alpha, \beta) = 1$ if and only if $\alpha \sim \beta$. Therefore, distance-regular networks become metric spaces with the distance function ∂.

In a distance-regular network, we have $p^i_{j1} = 0$ (for $i \neq 0$, j dose not belong to $\{i-1, i, i+1\}$), i.e., the non-zero intersection numbers of the network are given by

$$a_i = p^i_{i1}, \quad b_i = p^i_{i+1,1}, \quad c_i = p^i_{i-1,1} \ , \tag{2-7}$$

respectively (for more details see [1]). The intersection numbers (2-7) and the valencies κ_i satisfy the following obvious conditions

$$a_i + b_i + c_i = \kappa, \quad \kappa_{i-1} b_{i-1} = \kappa_i c_i, \quad i = 1, ..., d,$$
$$\kappa_0 = c_1 = 1, \quad b_0 = \kappa_1 = \kappa, \quad (c_0 = b_d = 0). \tag{2-8}$$

Thus all parameters of the network can be obtained from the intersection array $\{b_0, ..., b_{d-1}; c_1, ..., c_d\}$.

By using the equations (2-2) and (2-8), for adjacency matrices of distance-regular network Γ, we obtain

$$A_1 A_i = b_{i-1} A_{i-1} + (\kappa - b_i - c_i) A_i + c_{i+1} A_{i+1}, \quad i = 1, 2, ..., d-1,$$
$$A_1 A_d = b_{d-1} A_{d-1} + (\kappa - c_d) A_d. \tag{2-9}$$

The recursion relations (2-9), imply that

$$A_i = P_i(A), \quad i = 0, 1, ..., d. \tag{2-10}$$

2.1 Stratification

For an underlying network Γ, let W denotes the vector space over C consisting of column vectors whose coordinates are indexed by vertex set V of Γ, and whose entries are in C (i.e., $W = C^N$, with $N = |V|$). We observe that all $N \times N$ matrices with entries from C act on W by left multiplication. We endow W with the Hermitian inner product $\langle , \rangle$ which satisfies $\langle u, v\rangle = u^t\bar{v}$ for all $u, v \in W$, where t denotes the transpose and - denotes the complex conjugation. For all $\beta \in V$, let $|\beta\rangle$ denote the element of W with a 1 in the β coordinate and 0 in all other coordinates. We observe $\{|\beta\rangle | \beta \in V\}$ is an orthonormal basis for W, but in this basis, W is reducible and can be reduced to irreducible subspaces W_i, $i = 0, 1, ..., d$, i.e.,

$$W = W_0 \oplus W_1 \oplus ... \oplus W_d, \tag{2-11}$$

where, d is diameter of the corresponding association scheme. In the following we introduce orthonormal basis for irreducible subspace of W with maximal dimension, explicitly. To do so, for a given vertex $\alpha \in V$, we define $\Gamma_i(\alpha) = \{\beta \in V : (\alpha, \beta) \in R_i\}$. Then, the vertex set V can be written as disjoint union of $\Gamma_i(\alpha)$, i.e.,

$$V = \bigcup_{i=0}^{d} \Gamma_i(\alpha). \tag{2-12}$$

Now, we fix a point $o \in V$ as an origin of the underlying network, called reference vertex. Then, the relation (2-12) stratifies the network into a disjoint union of strata (associate classes) $\Gamma_i(o)$. With each stratum $\Gamma_i(o)$ we associate a unit vector $|\phi_i\rangle$ in W (called unit vector of i-th stratum) defined by

$$|\phi_i\rangle = \frac{1}{\sqrt{\kappa_i}} \sum_{\alpha \in \Gamma_i(o)} |\alpha\rangle, \tag{2-13}$$

where, $|\alpha\rangle$ denotes the eigenket of α-th vertex at the associate class $\Gamma_i(o)$ and $\kappa_i = |\Gamma_i(o)|$ is called the i-th valency of the network ($\kappa_i := p^0_{ii} = |\{\gamma : (o, \gamma) \in R_i\}| = |\Gamma_i(o)|$). For $0 \leq i \leq d$, the unit vectors $|\phi_i\rangle$ of Eq.(2-13) form an orthonormal basis for irreducible submodule of W with maximal dimension denoted by W_0.

Let A_i be the ith adjacency matrix of the underlying network Γ. From the action of A_i on reference state $|\phi_0\rangle$ ($|\phi_0\rangle = |o\rangle$, with $o \in V$ as reference vertex), we have

$$A_i|\phi_0\rangle = \sum_{\beta\in\Gamma_i(o)} |\beta\rangle. \tag{2-14}$$

Then by using (2-13) and (2-14), we obtain

$$A_i|\phi_0\rangle = \sqrt{\kappa_i}|\phi_i\rangle. \tag{2-15}$$

By using (2-15), we can write

$$A_i|\phi_j\rangle = \frac{1}{\sqrt{\kappa_j}}A_iA_j|\phi_0\rangle = \frac{1}{\sqrt{\kappa_j}}\sum_k p_{ij}^k A_k|\phi_0\rangle = \frac{1}{\sqrt{\kappa_j}}\sum_k \sqrt{\kappa_k}p_{ij}^k|\phi_k\rangle. \tag{2-16}$$

Therefore, from the orthonormality of unit vectors $|\phi_i\rangle$, for $i = 0, 1, ..., d$, we obtain

$$\langle\phi_l|A_i|\phi_j\rangle = \sqrt{\frac{\kappa_l}{\kappa_j}}p_{ij}^l. \tag{2-17}$$

It could be noticed that, in the case of distance-regular networks, the adjacency matrices A_i are tridiagonal in the basis of $|\phi_i\rangle$ (see [10], for more details).

3 Effective resistances on resistor networks

3.1 General networks

For a given regular network Γ with N vertices and adjacency matrix A, let $r_{ij} = r_{ji}$ be the resistance of the resistor connecting vertices i and j. Hence, the conductance is $c_{ij} = r_{ij}^{-1} = c_{ji}$ so that $c_{ij} = 0$ if there is no resistor connecting i and j. Denote the electric potential at the i-th vertex by V_i and the net current flowing into the network at the i-th vertex by I_i (which is zero if the i-th vertex is not connected to the external world). Since there exist no sinks or sources of current including the external world, we have the constraint $\sum_{i=1}^{N} I_i = 0$. The Kirchhoff law states

$$\sum_{j=1, j\neq i}^{N} c_{ij}(V_i - V_j) = I_i, \quad i = 1, 2, ..., N. \tag{3-18}$$

Explicitly, Eq.(3-18) reads

$$L\vec{V} = \vec{I}, \tag{3-19}$$

where, $\vec{V}$ and $\vec{I}$ are n-vectors whose components are V_i and I_i, respectively and

$$L = \sum_{i=1}^{N} c_i |i\rangle\langle i| - \sum_{i,j=1}^{N} c_{ij} |i\rangle\langle j| \tag{3-20}$$

is the Laplacian of the graph Γ with

$$c_i \equiv \sum_{j=1, j\neq i}^{N} c_{ij}, \tag{3-21}$$

for each vertex α. It should be noticed that, L has eigenvector $(1,1,...,1)^t$ with eigenvalue 0. Therefore, L is not invertible and so we define the psudo-inverse of L as

$$L^{-1} = \sum_{i,\lambda_i\neq 0} \lambda_i^{-1} E_i, \tag{3-22}$$

where, E_i is the operator of projection onto the eigenspace of L^{-1} corresponding to eigenvalue λ_i. It has been shown that, the effective resistances $R_{\alpha\beta}$ are given by

$$R_{\alpha\beta} = \langle\alpha|L^{-1}|\alpha\rangle + \langle\beta|L^{-1}|\beta\rangle - \langle\alpha|L^{-1}|\beta\rangle - \langle\beta|L^{-1}|\alpha\rangle. \tag{3-23}$$

This formula may be formally derived [19] using Kirchoff 's laws, and seems to have been long known in the electrical engineering literature, with it appearing in several texts, such as Ref. [21].

3.2 Underlying resistor networks of association schemes

In the present paper we deal with special networks which are underlying networks of some symmetric association schemes. For these networks, first we choose a vertex, say α, as reference vertex and stratify the network with respect to α. Then, we assume that the conductance between α and β is c_i for all $\beta \in \Gamma_i(\alpha)$, i.e., the conductances between α and all vertices

belonging to the same strata (with respect to α) are the same. Then, the Laplacian of the underlying network is defined as

$$L = (\sum_{i=0}^{d} c_i\kappa_i)I - \sum_{i=0}^{d} c_i A_i, \tag{3-24}$$

where, d is the diameter of the association scheme. Obviously, in the case that all nonzero resistances are connecting resistances and are equal to 1 ($c \equiv c_1 = 1$, $c_i = 0$ for $i \neq 1$), the off-diagonal elements of $-L$ are precisely those of the adjacency matrix A of the network, i.e.,

$$L = \kappa I - A, \tag{3-25}$$

where, $\kappa \equiv \kappa_1 = deg(\alpha)$ (in regular networks, the degree is independent of the vertex α). In section 5, we will show that in the case of underlying networks of association schemes, all of the diagonal entries of the pseudo inverse matrix L^{-1} are equal. By using this result and from the fact that L^{-1} is a real matrix, the Eq.(3-23) can be written for these networks as follows

$$R_{\alpha\beta} = 2(\langle\alpha|L^{-1}|\alpha\rangle - \langle\alpha|L^{-1}|\beta\rangle). \tag{3-26}$$

4 Calculating effective resistances on underlying networks of association schemes without using the spectrum of the networks

In general, as it will be shown in the next section (the Eq.(5-72)), in order to calculate the effective resistances on underlying networks of association schemes, one needs to know the spectrum of the adjacency matrices A_i for $i = 1, ..., d$. Although, in the most cases the spectrum of the Bose-Mesner algebra is known, but the formula for effective resistances in terms of the spectrum of the network do not possess a closed form and evaluation of it in the most cases is not an easy task. In this section, we assume that all of the conductances

except for one of them, say $c \equiv c_1 = 1$, are zero and consider particular underlying networks of association schemes such that the adjacency matrices A_i are written as polynomials of the first adjacency matrix $A = A_1$ (not necessarily of degree i). Then, we give a procedure for evaluating the effective resistances on these networks such that one can calculate the effective resistances by using the structure of their Bose-Mesner algebra without any need to know the spectrum of the adjacency matrices (even if there is no any three-term recursion relations such as (2-9) which are satisfied by distance-regular networks). In the appendix A, we show that if the adjacency matrix of a connected underlying network of association scheme with diameter d possesses $d+1$ distinct eigenvalues, then all of the other adjacency matrices A_i for $i = 2, ..., d$ can be written as polynomials of A.

As it will be shown in section 5 (Corollary 1), all of the nodes β belonging to the same stratum with respect to the reference node α, possess the same effective resistance with respect to α. This allows us to write

$$R_{\alpha\beta^{(m)}} = \frac{1}{\kappa_m} \sum_{\beta \in \Gamma_m(\alpha)} R_{\alpha\beta} = \frac{1}{\kappa_m} \sum_{\beta \in V} (A_m)_{\alpha\beta} R_{\alpha\beta}.$$

where, $R_{\alpha\beta^{(m)}}$ denotes the effective resistances between α and all of the nodes $\beta \in \Gamma_m(\alpha)$. Now, consider underlying networks of association schemes for which we have $A_m = \sum_{n=0}^{d} c_{mn} A^n$ (recall that, for underlying networks which satisfy the distance-regularity condition, i.e., the three-term recursion relations (2-9) are satisfied, we have $A_m = P_m(A) = \sum_{n=0}^{m} c_{mn} A^n$, where P_m is a polynomial of degree m). Then, by using (3-26) one can obtain

$$R_{\alpha\beta^{(m)}} = \frac{2}{\kappa_m} \sum_{n=0}^{d} c_{mn} [\sum_{\beta \in V} (A^n)_{\alpha\beta} L^{-1}_{\alpha\alpha} - \sum_{\beta \in V} (A^n)_{\alpha\beta} L^{-1}_{\alpha\beta}]. \qquad (4\text{-}27)$$

From the fact that, the effective resistances $R_{\alpha\beta^{(m)}}$ are independent of the choice of the reference node α, one can write

$$\sum_{\alpha \in V} R_{\alpha\beta^{(m)}} = N.R_{\alpha\beta^{(m)}} = \frac{2}{\kappa_m} \sum_{n=0}^{d} c_{mn} [\sum_{\alpha,\beta \in V} (A^n)_{\alpha\beta} L^{-1}_{\alpha\alpha} - \sum_{\alpha,\beta \in V} (A^n)_{\alpha\beta} L^{-1}_{\alpha\beta}]. \qquad (4\text{-}28)$$

Now, we note that

$$\sum_{\beta\in V}(A^n)_{\alpha\beta}=\sum_{\beta,\gamma_1,...,\gamma_{n-1}\in V}A_{\alpha\gamma_1}A_{\gamma_1\gamma_2}...A_{\gamma_{n-1}\beta}=\kappa^n.$$

Now, assume that all of the conductances c_i equal to zero except for $c_1\equiv c=1$ which implies that the Eq. (3-25) is satisfied. Then, from (4-28), we obtain

$$R_{\alpha\beta^{(m)}}=\frac{2}{N.\kappa_m}\sum_{n=0}^{d}c_{mn}[\kappa^n.tr(L^{-1})-tr(A^nL^{-1})]=\frac{2}{N.\kappa_m}\sum_{n=0}^{d}c_{mn}tr(\frac{\kappa^n\mathbf{1}-A^n}{\kappa\mathbf{1}-A})=$$

$$\frac{2}{N.\kappa_m}\sum_{n=1}^{d}c_{mn}tr[(\kappa^{n-1}\mathbf{1}+\kappa^{n-2}A+\kappa^{n-3}A^2+...+A^{n-1})(\mathbf{1}-1/NJ)]. \tag{4-29}$$

By using the equality $A^lJ=\kappa^lJ$ (recall that $AJ=\kappa J$), the Eq.(4-29) can be rewritten as follows

$$R_{\alpha\beta^{(m)}}=\frac{2}{N.\kappa_m}\sum_{n=1}^{d}c_{mn}(\sum_{i=1}^{n}\kappa^{n-i}tr(A^{i-1})-n.\kappa^{n-1}). \tag{4-30}$$

As the above formula indicates, in order to calculate $R_{\alpha\beta^{(m)}}$, we need to evaluate $tr(A^l)$, for all $l=1,2,...,d-1$. To this aim, we use the relations $A_m=\sum_{n=0}^{d}c_{mn}A^n$ for $m=1,...,d$ to write A^l as $A^l=\sum_{m=0}^{d}c'_{lm}A_m$ and obtain $tr(A^l)=N.c'_{l0}$.

4.1 Examples

1. Underlying network of association scheme derived from $Z_5\times Z_5$

In the regular representation, the elements of abelian group $Z_5\times Z_5$ are written as $S_1^kS_2^l$ with $S_1=S\otimes I$ and $S_2=I\otimes S$, where S is the shift operator with period 5, i.e., $S^5=I_5$. Now, we define the following adjacency matrices

$$A\equiv A_1=S_1+S_2+S_1S_2+S_1^4+S_2^4+S_1^4S_2^4,$$

$$A_2=S_1^2+S_2^2+S_1^2S_2^2+S_1^3+S_2^3+S_1^3S_2^3,$$

$$A_3=S_1^3+S_2^3+S_1^3S_2^4+S_1^4S_2^3+S_1^2S_2+S_1S_2^2,$$

$$A_4=S_1S_2^3+S_1^3S_2+S_1^2S_2^3+S_1^3S_2^2+S_1^2S_2^4+S_1^4S_2^2.$$

Then, one can easily see that the above adjacency matrices constitute the Bose-Mesner algebra of a symmetric association scheme. In fact, we have

$$A^2 = 6A_0+2A+A_2+2A_3, \quad AA_2 = A+A_2+2A_3+2A_4, \quad AA_3 = 2A+2A_2+2A_4, \quad AA_4 = 2A_2+2A_3+2A_4. \tag{4-31}$$

The above relations indicate that the underlying network of the constructed association scheme is not distance regular. By using (4-31), one can evaluate the powers of A as follows

$$A^2 = 6A_0+2A+A_2+2A_3, \quad A^3 = 12A_0+15A+7A_2+6A_3+6A_4, \quad A^4 = 90A_0+61A+46A_2+56A_3+38A_4. \tag{4-32}$$

Then, by solving (4-32) in terms of A_2, A_3 and A_4, we obtain

$$A_2 = \frac{1}{44}(-6A^4+38A^3+54A^2-312A-240A_0), \quad A_3 = \frac{1}{44}(3A^4-19A^3-5A^2+112A-12A_0),$$
$$A_4 = \frac{1}{22}(2A^4+9A^3-29A^2+71A+102A_0). \tag{4-33}$$

That is, the coefficients c_{mn} in $A_m = \sum_{n=0}^{d} c_{mn}A^n$ are given by

$$c_{11}=1, \;\; c_{1i}=0 \;\text{ for }\; i\neq 1; \quad c_{20}=\frac{60}{11}, \;\; c_{21}=-\frac{78}{11}, \;\; c_{22}=\frac{27}{22}, \;\; c_{23}=\frac{19}{22}, \;\; c_{24}=-\frac{3}{22}; \;\; c_{30}=-\frac{3}{11},$$
$$c_{31}=\frac{28}{11}, \;\; c_{32}=-\frac{5}{44}, \;\; c_{33}=-\frac{19}{44}, \;\; c_{34}=\frac{3}{44}; \;\; c_{40}=\frac{51}{11}, \;\; c_{41}=\frac{71}{22}, \;\; c_{42}=-\frac{29}{22}, \;\; c_{43}=\frac{9}{22}, \;\; c_{44}=\frac{1}{11}. \tag{4-34}$$

Then, by using (4-32), (4-34) and substituting $N=25$ and $\kappa=\kappa_2=\kappa_3=\kappa_4=6$ in the result (4-30), we obtain the effective resistances as follows:

$$R_{\alpha\beta^{(1)}} = \frac{1}{75}c_{11}(25-1) = \frac{24}{75},$$
$$R_{\alpha\beta^{(2)}} = \frac{1}{75}\{24c_{21}+138c_{22}+942c_{23}+5736c_{24}) = \frac{112}{275},$$
$$R_{\alpha\beta^{(3)}} = \frac{1}{75}\{24c_{21}+138c_{22}+942c_{23}+5736c_{24}) = \frac{327}{825},$$
$$R_{\alpha\beta^{(4)}} = \frac{1}{75}\{24c_{21}+138c_{22}+942c_{23}+5736c_{24}) = \frac{2942}{275}. \tag{4-35}$$

2. Group association scheme S_4

In group association schemes, the adjacency matrices are defined as the class sums of a group in regular representation. For instance, in the symmetric group S_4, the conjugacy classes are given by

$$C_0 = \{1\},\ \ C_1 = \{(12),(13),(14),(23),(24),(34)\},\ \ C_2 = \{(123),(132),(124),(142),(134),(143),(234),$$

$$(243)\},\ \ C_3 = \{(12)(34),(13)(24),(14)(23)\},\ \ C_4 = \{(1234),(1243),(1324),(1342),(1423),(1432)\}. \qquad (4\text{-}36)$$

Then, the adjacency matrices are defined as $A_i = \bar{C}_i, \ \ i = 0,1,...,4$, i.e., we have

$$A \equiv A_1 = (12)+(13)+(14)+(23)+(24)+(34), \quad A_2 = (123)+(132)+(124)+(142)+(134)+(143)+(234)+$$

$$(243),\ \ A_3 = (12)(34)+(13)(24)+(14)(23),\ \ A_4 = (1234)+(1243)+(1324)+(1342)+(1423)+(1432).$$

One can easily show that these adjacency matrices satisfy the following relations

$$A^2 = 6A_0 + 3A_2 + 2A_3, \quad AA_2 = 4A + 4A_4, \quad AA_3 = A + 2A_4, \quad AA_4 = 4A_2 + 4A_3, \quad (4\text{-}37)$$

above relations indicate that the group association scheme S_4 is not distance regular (actually group scheme S_n is a distance-regular one, only for $n = 3$). By using (4-37), one can evaluate the powers of A as follows

$$A^2 = 6A_0 + 3A_2 + 2A_3, \quad A^3 = 20A + 16A_4, \quad A^4 = 120A_0 + 108A_2 + 104A_3. \qquad (4\text{-}38)$$

Then, by solving (4-38) in terms of A_2, A_3 and A_4, we obtain

$$A_2 = -\frac{1}{48}(A^4 - 52A^2 + 192A_0), \quad A_3 = \frac{1}{32}(A^4 - 36A^2 + 96A_0), \quad A_4 = \frac{1}{16}(A^3 - 20A). \ (4\text{-}39)$$

That is, the coefficients c_{mn} in $A_m = \sum_{n=0}^{d} c_{mn}A^n$ are given by

$$c_{11} = 1,\ \ c_{1i} = 0 \ \text{ for } \ i \neq 1; \quad c_{20} = -4,\ \ c_{21} = 0,\ \ c_{22} = \frac{13}{12},\ \ c_{23} = 0,\ \ c_{24} = -\frac{1}{48};\ \ c_{30} = 3,$$

$$c_{31} = 0, \quad c_{32} = -\frac{9}{8}, \quad c_{33} = 0, \quad c_{34} = \frac{1}{32}; \quad c_{40} = 0, \quad c_{41} = -\frac{5}{4}, \quad c_{42} = 0, \quad c_{43} = \frac{1}{16}, \quad c_{44} = 0. \tag{4-40}$$

Then, by using (4-38), (4-40) and substituting $N = 24$ and $\kappa = 6, \quad \kappa_2 = 8, \quad \kappa_3 = 3, \quad \kappa_4 = 6$ in the result (4-30), we obtain the effective resistances as follows:

$$R_{\alpha\beta^{(1)}} = \frac{1}{72}c_{11}(24-1) = \frac{23}{72}, \qquad R_{\alpha\beta^{(2)}} = \frac{1}{96}\{132c_{22} + 6048c_{24}) = \frac{35}{96},$$
$$R_{\alpha\beta^{(3)}} = \frac{1}{36}\{132c_{32} + 5184c_{34}\} = \frac{3}{8}, \qquad R_{\alpha\beta^{(4)}} = \frac{1}{72}\{23c_{41} + 1620c_{43} = \frac{145}{36}. \tag{4-41}$$

In order to give another nontrivial examples of underlying networks of association schemes which are not distance-regular networks, we construct two association schemes with diameter 6 by combining the class sums of the symmetric group S_4 as follows:

a) We define the adjacency matrices as follows:

$$A_0 = I, \;\; A \equiv A_1 = (12)+(13)+(14), \;\; A_2 = (123)+(132)+(124)+(142)+(134)+(143), \;\; A_3 = (23)+(24)+(34),$$

$$A_4 = (1234)+(1243)+(1324)+(1342)+(1423)+(1432), \quad A_5 = (12)(34)+(13)(24)+(14)(23), \;\; A_6 = (234)+(243). \tag{4-42}$$

Then, one can show that the following relations are satisfied

$$A^2 = 3A_0 + A_2, \quad AA_2 = 2A + 2A_3 + A_4, \quad AA_3 = A_2 + A_5,$$

$$AA_5 = A_3 + A_4, \quad AA_4 = A_2 + 2A_5 + 3A_6, \quad AA_6 = A_4. \tag{4-43}$$

The relations (4-43) indicate that the underlying network is not distance-regular (see Eq. (2-9)). Now, in order to evaluate the effective resistances on this network, we calculate the powers of the adjacency matrix A as follows:

$$A^2 = 3A_0 + A_2, \quad A^3 = 5A + 2A_3 + A_4, \quad A^4 = 15A_0 + 8A_2 + 4A_5 + 3A_6,$$

$$A^5 = 31A + 20A_3 + 15A_4, \quad A^6 = 93A_0 + 66A_2 + 50A_5 + 45A_6. \tag{4-44}$$

Then by solving the five equations with five unknown $A_2, ..., A_6$, we obtain the following solution

$$A_2 = A^2 - 3A_0, \quad A_3 = \frac{1}{10}(-A^5 + 15A^3 - 44A), \quad A_4 = \frac{1}{5}(A^5 - 10A^3 + 19A),$$
$$A_5 = \frac{1}{10}(-A^6 + 15A^4 - 54A^2 + 30A_0), \quad A_6 = \frac{1}{15}(2A^6 - 25A^4 + 68A^2 - 15A_0). \qquad (4\text{-}45)$$

That is, the coefficients c_{mn} in $A_m = \sum_{n=0}^{d} c_{mn}A^n$ are given by

$$c_{11} = 1, \quad c_{1i} = 0 \ \text{ for } \ i \neq 1; \quad c_{20} = -3, \quad c_{22} = 1, \quad c_{21} = c_{23} = c_{24} = c_{25} = c_{26} = 0;$$
$$c_{30} = c_{32} = c_{34} = c_{36} = 0, \quad c_{31} = -\frac{22}{5}, \quad c_{33} = \frac{3}{2}, \quad c_{35} = -\frac{1}{10}; \quad c_{40} = c_{42} = c_{44} = c_{46} = 0,$$
$$c_{41} = \frac{19}{5}, c_{43} = -2, \quad c_{45} = \frac{1}{5}; \quad c_{50} = 3, \quad c_{51} = c_{53} = c_{55} = 0, \quad c_{52} = -\frac{27}{5}, \quad c_{54} = \frac{3}{2}, \quad c_{56} = -\frac{1}{10};$$
$$c_{60} = -1, \quad c_{61} = c_{63} = c_{65} = 0, \quad c_{62} = \frac{68}{15}, \quad c_{64} = -\frac{5}{3}, \quad c_{66} = \frac{2}{15}. \qquad (4\text{-}46)$$

Then, by using (4-45), (4-46) and substituting $N = 24$ and $\kappa = 3, \quad \kappa_2 = 3, \quad \kappa_3 = 6, \quad \kappa_4 = 2, \quad \kappa_5 = 3, \quad \kappa_6 = 6$ in the result (4-30), we obtain the effective resistances as follows:

$$R_{\alpha\beta^{(1)}} = \frac{1}{36}c_{11}(24-1) = \frac{23}{36}, \quad R_{\alpha\beta^{(2)}} = \frac{1}{72}c_{22}(72-6) = \frac{33}{36},$$
$$R_{\alpha\beta^{(3)}} = \frac{1}{36}\{c_{31}(24-1) + c_{33}(\sum_{i=1}^{3} 3^{3-i}tr(A^{i-1}) - 27) + c_{35}(\sum_{i=1}^{5} 3^{5-i}tr(A^{i-1}) - 405)\} = \frac{89}{90},$$
$$R_{\alpha\beta^{(4)}} = \frac{1}{36}\{c_{41}(24-1) + c_{43}(\sum_{i=1}^{3} 3^{3-i}tr(A^{i-1}) - 27) + c_{45}(\sum_{i=1}^{5} 3^{5-i}tr(A^{i-1}) - 405)\} = \frac{187}{180},$$
$$R_{\alpha\beta^{(5)}} = \frac{1}{36}\{c_{52}(72-6) + c_{54}(\sum_{i=1}^{4} 3^{4-i}tr(A^{i-1}) - 108) + c_{56}(\sum_{i=1}^{6} 3^{6-i}tr(A^{i-1}) - 1458)\} = \frac{21}{20},$$
$$R_{\alpha\beta^{(6)}} = \frac{1}{24}\{c_{62}(72-6) + c_{64}(\sum_{i=1}^{4} 3^{4-i}tr(A^{i-1}) - 108) + c_{66}(\sum_{i=1}^{6} 3^{6-i}tr(A^{i-1}) - 1458)\} = \frac{16}{15}. \qquad (4\text{-}47)$$

b) If we choose the matrix A_4 in (4-42) as adjacency matrix, we obtain another connected network (the matrices A_2, A_3, A_5 and A_6 do not define connected networks). Then, one can obtain

$$A_4A_1 = A_2 + 2A_5 + 3A_6, \quad A_4A_2 = 2A_1 + 4A_3 + 3A_4, \quad A_4A_3 = 2A_2 + 2A_5,$$

$$A_4^2 = 6A_0 + 3A_2 + 2A_5 + 3A_6, \quad A_4A_5 = 2A_1 + 2A_3 + A_4, \quad A_4A_6 = 2A_1 + A_4. \qquad (4\text{-}48)$$

which indicate that the underlying network is not distance-regular (see Eq. (2-9)). Again, in order to evaluate the effective resistances on this network, we calculate the powers of the adjacency matrix $A \equiv A_4$ as follows:

$$A^2 = 6A_0 + 3A_3 + 3A + 2A_5, \quad A^3 = 16(A_1 + A_2) + 20A_6, \quad A^4 = 120A_0 + 108(A_3 + A) + 104A_5,$$

$$A^5 = 640(A_1 + A_2) + 656A_6, \quad A^6 = 3936A_0 + 3888(A_3 + A) + 3872A_5. \qquad (4\text{-}49)$$

By solving (4-49), one can write A_i for $i = 1, 2, 3, 5, 6$ in terms of powers of $A \equiv A_4$ and (similar to the case a) evaluate effective resistances $R_{\alpha\beta^{(i)}}$ for $i = 1, 2, 3, 5, 6$.

It should be noticed that, in distance-regular networks, by using the three-term recursion relations (2-9), one can obtain

$$A^2 = AA_1 = \kappa\mathbf{1} + a_1A + c_2A_2,$$

$$A^3 = AA^2 = \kappa a_1\mathbf{1} + (\kappa + a_1^2 + b_1c_2)A + (a_1 + a_2)c_2A_2 + c_2c_3A_3,$$

$$A^4 = AA^3 = I_0\mathbf{1} + I_1A + I_2A_2 + I_3A_3 + c_2c_3c_4A_4,$$

$$A^5 = AA^4 = \kappa I_1\mathbf{1} + (I_0 + a_1I_1 + b_1I_2)A + (c_2I_1 + a_2I_2 + b_2I_3)A_3 + (c_3I_2 + a_3I_3 + b_3c_2c_3c_4)A_3+$$

$$(c_4I_3 + a_4c_2c_3c_4)A_4 + c_2c_3c_4c_5A_5, \qquad (4\text{-}50)$$

where,

$$I_0 := \kappa(\kappa+a_1^2+b_1c_2), \quad I_1 := a_1(2\kappa+a_1^2+2b_1c_2)+b_1c_2a_2, \quad I_2 := c_2[\kappa+a_1^2+b_1c_2+a_2(a_1+a_2)+b_2c_3],$$

$$I_3 := c_2c_3(a_1 + a_2 + a_3). \qquad (4\text{-}51)$$

Therefore, by using (4-30) and (4-50), for distance-regular resistor networks such that all of the conductances c_i are equal to zero except for one of them, i.e., $c_1 \equiv c = 1$ and $c_i = 0$ for $i \neq 1$, the effective resistances $R_{\alpha\beta^{(m)}}$ for $m = 1, 2, ..., 5$ are obtained in terms of the intersection numbers of the network as follows

$$R_{\alpha\beta^{(1)}} = \frac{2}{\kappa}(\frac{N-1}{N}),$$

$$R_{\alpha\beta^{(2)}} = \frac{2}{\kappa b_1}\{b_1 + 1 - \frac{\kappa + b_1 + 1}{N}\},$$
$$R_{\alpha\beta^{(3)}} = \frac{2}{\kappa b_1 b_2}\{b_{d-1}c_d + b_2 - \kappa + c_2 + b_1 b_2 - \frac{(\kappa+1)(b_2+c_2) + b_1(\kappa + b_2)}{N}\},$$
$$R_{\alpha\beta^{(4)}} = \frac{2}{\kappa b_1 b_2 b_3}\{-I_1(1-\frac{1}{N}) - \kappa I_2(1-\frac{2}{N}) - \kappa I_3(\kappa + 1 - 3\frac{\kappa}{N}) + \kappa^3(1-\frac{4}{N}) + \kappa.(\kappa + a_1)\},$$
$$R_{\alpha\beta^{(5)}} = \frac{2}{\kappa b_1 b_2 b_3 b_4}\{-(I_0+a_1I_1+b_1I_2)(1-\frac{1}{N})-(c_2I_1+a_2I_2+b_2I_3)(\kappa-2)-(c_3I_2+a_3I_3+b_3c_2c_3c_4).$$
$$(\kappa^2(1-\frac{3}{N})+\kappa)-(c_4I_3+a_4c_2c_3c_4)(\kappa^3(1-\frac{4}{N})+\kappa(\kappa+a_1))+\kappa^4(1-\frac{5}{N})+\kappa(\kappa^2+\kappa a_1+\kappa+a_1^2+b_1c_2)\}. \tag{4-52}$$

In Ref. [1], analytical formulas for effective resistances up to the third stratum, i.e., $R_{\alpha\beta^{(i)}}$ for $i = 1, 2, 3$ on distance-regular networks have been given by using the stieltjes function associated with the network, where the results are in agreement with (4-52). It should be noticed that, by using the above result, one can evaluate the limiting value of the effective resistances in the limit of the large size of the networks, i.e., in the limit $N \to \infty$. For instance, the effective resistances $R_{\alpha\beta^{(1)}}$, $R_{\alpha\beta^{(2)}}$ and $R_{\alpha\beta^{(3)}}$ tend to $\frac{2}{\kappa}$, $\frac{2(b_1+1)}{\kappa b_1}$ and $\frac{2(b_{d-1}c_d+b_2-\kappa+c_2+b_1b_2)}{\kappa b_1 b_2}$, respectively which are finite for the resistor networks with finite value of the valency κ.

In the following, we introduce some interesting distance-regular networks which are underlying networks of association schemes derived from symmetric group S_n and its nontrivial subgroups and calculate the effective resistances on these networks.

4.1.1 Association schemes derived from symmetric group S_n

Let $\lambda = (\lambda_1, ..., \lambda_m)$ be a partition of n, i.e., $\lambda_1 + ... + \lambda_m = n$. We consider the subgroup $S_m \otimes S_{n-m}$ of S_n with $m \leq [\frac{n}{2}]$. Then we assume the finite set M^λ (where, the association scheme is defined on it) as $M^\lambda = \frac{S_n}{S_m \otimes S_{n-m}}$ with $|M^\lambda| = \frac{n!}{m!(n-m)!}$. In fact, M^λ is the set of $(m-1)$-faces of $(n-1)$-simplex (note that, the graph of an $(n-1)$-simplex is the complete graph with n vertices denoted by K_n). If we denote the vertex i by m-tuple $(i_1, i_2, ..., i_m)$, then the relations R_k, $k = 0, 1, ..., m$ defined by

$$R_k = \{(i,j) : \partial(i,j) = k\}, \quad k = 0, 1, ..., m, \tag{4-53}$$

where, we mean by $\partial(i,j)$ the number of components that $i=(i_1,i_2,...,i_m)$ and $j=(j_1,j_2,...,j_m)$ are different (this is the same as Hamming distance which is defined in coding theory). It can be shown that, the relations R_k, for $k=0,1,...,m$ define an association scheme on $\frac{S_n}{S_m\otimes S_{n-m}}$ with diameter $m+1$, where the adjacency matrices A_k, $k=0,1,...,m$ are defined as

$$\big(A_k\big)_{i,j} = \begin{cases} 1 \quad \text{if} \quad \partial(i,j)=k, & \\ 0 \qquad \text{otherwise} \qquad\qquad (i,j\in M^\lambda) \end{cases}, \quad k=0,1,...,m. \tag{4-54}$$

One should notice that, the representation space M^λ is a module space which is not irreducible, i.e., M^λ is decomposed as

$$M^\lambda \cong \bigoplus_{\mu\trianglelefteq\lambda} S^\mu, \tag{4-55}$$

where, $\mu\trianglelefteq\lambda$ means that $\mu_1+...+\mu_i\leq\lambda_1+...+\lambda_i$, for each $i=1,...,m$. The number of distinct irreducible submodules of M^λ is $m+1$. The irreducible submodules S^μ are called Specht modules, where S^λ is the same as permutation module. The $m+1$ idempotents are defined by

$$E_\mu = \frac{\chi_\mu(e)}{n!}\sum_{g\in S_n}\chi_\mu(g)\rho(g), \tag{4-56}$$

where, e is the identity element, χ_μ is the character corresponding to the irreducible submodule S^μ and ρ is the representation of S_n over M^λ.

For the network with adjacency matrices defined by (4-54), one can show that the sizes of strata (valencies) are given by

$$\kappa_0=1, \quad \kappa_l = \begin{pmatrix} m \\ m-l \end{pmatrix}\begin{pmatrix} n-m \\ l \end{pmatrix}, \quad l=1,2,...,m \tag{4-57}$$

(clearly we have $N=\sum_{l=0}^m\kappa_l=\begin{pmatrix} n \\ m \end{pmatrix}=\frac{n!}{m!(n-m)!}=|M^\lambda|$). If we stratify the network with respect to reference node $|\phi_0\rangle=|i_1,i_2,...,i_m\rangle$, the unit vectors $|\phi_i\rangle$, $i=1,...,m$ are defined as

$$|\phi_1\rangle = \frac{1}{\sqrt{\kappa_1}}\Big(\sum_{i'_1\neq i_1}|i'_1,i_2,...,i_m\rangle+\sum_{i'_2\neq i_2}|i_1,i'_2,i_3,...,i_m\rangle+...+\sum_{i'_m\neq i_m}|i_1,...,i_{m-1},i'_m\rangle\Big),$$

$$|\phi_2\rangle = \frac{1}{\sqrt{\kappa_2}}\sum_{k\neq l=1}^{m}\sum_{i'_l\neq i_l;i'_k\neq i_k}|i_1,...i_{l-1},i'_l,i_{l+1},...,i_{k-1},i'_k,i_{k+1}...,i_m\rangle,$$

$$\vdots$$

$$|\phi_m\rangle = \frac{1}{\sqrt{\kappa_m}}\sum_{i'_1\neq i_1;...;i'_m\neq i_m}|i'_1,i'_2,...,i'_m\rangle. \tag{4-58}$$

The constructed network as in the above is a distance-regular network with intersection array as follows

$$b_l = (m-l)(n-m-l) \;\; ; \quad c_l = l^2. \tag{4-59}$$

Then, by using the Eq. (2-9), one can obtain

$$AA_l = (m-l+1)(n-m-l+1)A_{l-1} + l(n-2l)A_l + (l+1)^2A_{l+1}. \tag{4-60}$$

In the following, we consider the case $m=2$, where the vertices are edges of a complete graph K_n in details and calculate the effective resistances.

In the case of $m=2$, we have three kinds of relations as follows

$$R_0 = \{((ij),(ij))\}, \quad R_1 = \{((ij),(ik)),((ij),(kj)) : j\neq k, i\neq k\},$$

$$R_2 = \{((ij),(kl)) : i\neq k, j\neq l\}, \tag{4-61}$$

for $i<j=1,2,...,n$; $k<l=1,...,n$. Therefore, we have three adjacency matrices $A_0, A_1\equiv A$ and A_2, where $A_0 = I_{n(n-1)/2}$ and

$$(A)_{ij,kl} = \delta_{ik}(1-\delta_{jl}) + \delta_{jl}(1-\delta_{ik}),$$

$$(A_2)_{ij,kl} = (1-\delta_{ik})(1-\delta_{jl}), \quad i(k)<j(l)=1,2,...,n. \tag{4-62}$$

For a given vertex $|ij\rangle$, $i<j$ as reference vertex, the stratification basis $\{|\phi_i\rangle\}_{i=0,1,2}$ defined by (2-13), are obtained as

$$|\phi_0\rangle = |ij\rangle, \quad i<j=1,...,n,$$

$$|\phi_1\rangle = \frac{1}{\sqrt{2(n-2)}}(\sum_{k\neq j=1}^{n}|ik\rangle + \sum_{k\neq i=1}^{n}|kj\rangle),$$

$$|\phi_2\rangle = \frac{1}{\sqrt{\frac{(n-2)(n-3)}{2}}} \sum_{l\neq m=1\neq j\neq k}^{n} |lm\rangle. \tag{4-63}$$

Then, by using (4-62), one can obtain

$$A|\phi_0\rangle = \sqrt{2(n-2)}|\phi_1\rangle,$$

$$A|\phi_1\rangle = \sqrt{2(n-2)}|\phi_0\rangle + (n-2)|\phi_1\rangle + 2\sqrt{(n-3)}|\phi_2\rangle,$$

$$A|\phi_2\rangle = 2\sqrt{n-3}|\phi_1\rangle + 2(n-4)|\phi_2\rangle. \tag{4-64}$$

By using (4-57) and (4-59), we have

$$\kappa_0 = 1, \quad \kappa = \kappa_1 = 2(n-2), \quad \kappa_2 = \frac{(n-2)(n-3)}{2}; \quad \{b_0, b_1; c_1, c_2\} = \{2(n-2), n-3; 1, 4\}. \tag{4-65}$$

Then, by using the recursion relations (4-60), one can write

$$A^2 = 2(n-2).I_{\frac{n(n-1)}{2}} + (n-2).A + 4A_2,$$

$$AA_2 = (n-3)A + 2(n-4)A_2. \tag{4-66}$$

Now, by using the result (4-52), the effective resistances are evaluated as follows

$$R_{\alpha\beta^{(1)}} = \frac{n(n-1)-2}{n(n-1)(n-2)}, \quad R_{\alpha\beta^{(2)}} = \frac{n(n-1)+6}{n(n-1)(n-3)}. \tag{4-67}$$

5 Calculating effective resistances on underlying networks of association schemes by using the spectrum of the networks

In the following, we use the algebraic combinatoric structures of underlying resistor networks of association schemes in order to calculate effective resistances in (3-23) in terms of corresponding association scheme's parameters. By using (2-4), the Laplacian (3-24) can be written as

$$L = \sum_{k=0}^{d}(\sum_{i=0}^{d} c_i(\kappa_i - P_{ik}))E_k. \tag{5-68}$$

Then, we have

$$L^{-1}=\sum_{k=1}^{d}\frac{E_k}{\sum_{i=0}^{d}c_i(\kappa_i-P_{ik})} \tag{5-69}$$

Now, for each α and β, we consider α as reference vertex and $\beta\in\Gamma_l(\alpha)$. Then, the diagonal entries of L^{-1} are all the same and equal to

$$L^{-1}_{\alpha\alpha}=\sum_{k=1}^{d}\frac{\langle\alpha|E_k|\alpha\rangle}{\sum_{i=0}^{d}c_i(\kappa_i-P_{ik})}=\frac{1}{N}\sum_{k=1}^{d}\frac{m_k}{\sum_{i=0}^{d}c_i(\kappa_i-P_{ik})}, \tag{5-70}$$

where, we have used the fact that $\langle\alpha|E_k|\alpha\rangle=\frac{m_k}{N}$ with $m_k=rank(E_k)$. Also, we have

$$L^{-1}_{\beta\alpha}=\frac{1}{\sqrt{\kappa_l}}\langle\phi_l|L^{-1}|\alpha\rangle=\frac{1}{\kappa_l}\langle\alpha|A_lL^{-1}|\alpha\rangle=\frac{1}{\kappa_l}\langle\alpha|\sum_{k=1}^{d}\frac{A_lE_k}{\sum_{i=1}^{d}c_i(\kappa_i-P_{ik})}|\alpha\rangle$$

$$=\frac{1}{\kappa_l}\sum_{k=1}^{d}\frac{P_{lk}\langle\alpha|E_k|\alpha\rangle}{\sum_{i=1}^{d}c_i(\kappa_i-P_{ik})}=\frac{1}{N\kappa_l}\sum_{k=1}^{d}\frac{P_{lk}m_k}{\sum_{i=1}^{d}c_i(\kappa_i-P_{ik})}. \tag{5-71}$$

Therefore, by using (3-23), we obtain our main result as

$$R_{\alpha\beta^{(l)}}=\frac{2}{N\kappa_l}\sum_{k=1}^{d}\frac{m_k(\kappa_l-P_{lk})}{\sum_{i=1}^{d}c_i(\kappa_i-P_{ik})}\quad,\quad\forall\ \beta\in\Gamma_l(\alpha). \tag{5-72}$$

As the result (5-72) indicates, in order to calculate the effective resistances on underlying networks of association schemes, one needs to know the spectrum of the adjacency matrices A_i for $i=1,...,d$, i.e., one needs to know P_{il} for $i,l=0,...,d$. It should be also noticed that, in general the sums appearing in the Eq. (5-72) do not possess closed form and evaluation of them in the most cases is not an easy task. Despite of these problems, the result (5-72) leads us to some important facts. For instance, we conclude the following corollaries from the result (5-72):

Corollary 1 The effective resistances between a given node α and all of the nodes β belonging to the same strata with respect to α are the same.

Corollary 2 For the networks such that all of the conductances c_i are equal to zero except for one of them, i.e., $c_1\equiv c=1$ and $c_i=0$ for $i\neq 1$, we have

$$\sum_{\alpha,\beta;\beta\sim\alpha}R_{\alpha\beta}=\frac{N\kappa_1}{2}R_{\alpha\beta^{(1)}}=\sum_{k=1}^{d}\frac{m_k(\kappa_l-P_{lk})}{(\kappa_l-P_{lk})}=\sum_{k=1}^{d}m_k=N-1. \tag{5-73}$$

In fact, this particular result is true for any N-site connected network and was long ago established by Foster [22] and by Weinberg [23]. In Ref. [20], the sum in (5-73) has been identified as one of the sum rules of effective resistance as a metric (called resistance-distance) which is an invariant of the graph. It should be noticed that, the result (5-73) is special case of the following general result

Theorem For any N-site underlying network of association scheme, we have

$$S = 1/2\sum_{\alpha,\beta}\mathcal{A}_{\alpha\beta}R_{\alpha\beta} = N-1, \tag{5-74}$$

where, $\mathcal{A} := \sum_i c_i A_i$.

proof. By using the Eq.(3-26) and denoting $C = \sum_l c_l\kappa_l$, we have

$$S = 1/2\sum_{\alpha,\beta}\mathcal{A}_{\alpha\beta}R_{\alpha\beta} = \sum_{\alpha,\beta}\mathcal{A}_{\alpha\beta}(C\mathbf{1}-\mathcal{A})^{-1}_{\alpha\alpha} - \sum_{\alpha,\beta}\mathcal{A}_{\alpha\beta}(C\mathbf{1}-\mathcal{A})^{-1}_{\alpha\beta} =$$

$$\underbrace{\sum_l c_l\kappa_l}_{C}.tr(\frac{1}{C\mathbf{1}-\mathcal{A}}) - tr(\frac{\mathcal{A}}{C\mathbf{1}-\mathcal{A}}) = tr(\frac{C\mathbf{1}-\mathcal{A}}{C\mathbf{1}-\mathcal{A}}) = tr(\mathbf{1}-1/NJ) = N-1. \tag{5-75}$$

Corollary 3 By using the result (5-74), one can obtain a linear dependance between effective resistances as follows

$$c_1\kappa_1 R_{\alpha\beta^{(1)}} + c_2\kappa_2 R_{\alpha\beta^{(2)}} + ... + c_d\kappa_d R_{\alpha\beta^{(d)}} = \frac{2(N-1)}{N}. \tag{5-76}$$

proof. For a given α, the conductance between α and all of the nodes β which have the relation l with α ($\beta\in\Gamma_l(\alpha)$), is c_l (the number of such nodes β is equal to κ_l). Then, by using (5-74) one can write

$$\frac{1}{2}\sum_{\alpha,\beta}\mathcal{A}_{\alpha\beta}R_{\alpha\beta} = \frac{1}{2}N(c_1\kappa_1 R_{\alpha\beta^{(1)}} + c_2\kappa_2 R_{\alpha\beta^{(2)}} + ... + c_d\kappa_d R_{\alpha\beta^{(d)}}) = N-1. \tag{5-77}$$

Clearly, if only one of the conductances ($c_1 \equiv c$) be non zero, we will have

$$R_{\alpha\beta^{(1)}} = \frac{2(N-1)}{Nc\kappa_1}, \tag{5-78}$$

which is the same as the result (5-73).

One should notice that, for most underlying networks of association schemes the combinatorics structures of the networks such as m_i (the rank of the idempotents), κ_i (the valency of adjacency matrix A_i) and first eigenvalue matrix P, are known. For example, for all underlying networks of symmetric group association schemes, where the adjacency matrices are class sums of the group, these properties are easily evaluated. In fact, for these networks we have

$$m_i = d_i^2, \quad \kappa_i = \frac{|C_i|^2}{|G|}\sum_{\chi}\chi^2(g_i), \quad \text{with} \;\; g_i \in C_i, \quad P_{il} = \frac{\kappa_l}{d_i}\chi_i(\alpha_l), \tag{5-79}$$

where, χ_i is the character of the i-th irreducible representation of the group G, $d_i = \chi_i(0)$ and C_i is the i-th conjugacy class of G. Then, by using the Eq. (5-72), one can obtain

$$R_{\alpha\beta^{(l)}} = \frac{2}{|G|}\sum_{k=1}^{d}\frac{d_k(d_k - \chi_k(g_l))}{\sum_{i=1}^{d} c_i\kappa_i(1-\frac{\chi_k(g_i)}{d_k})} \quad , \qquad \forall \;\; \beta \in \Gamma_l(\alpha), \tag{5-80}$$

Although the result (5-72) can be applied to all underlying networks of association schemes such as distance-regular and strongly regular networks [11] and underlying networks of QD and GQD types [3, 10] where the corresponding combinatorics structures can be evaluated easily, in the following we will consider only special underlying networks of association schemes which we construct by using the orbits of the point groups corresponding to finite lattices such that in the limit of the large size of the lattices, we obtain root lattices of type A_n (for more details see Ref. [13]).

5.1 Examples

In this section, we consider some examples of the underlying networks of association schemes such as cycle network, infinite line network, hypercube network, finite and infinite square lattice and hexagonal network.

5.1.1 Cycle network C_{2k}

The cycle network C_N with $N = 2k$ vertices is a simple example which is underlying network of association scheme of class k, where all conductances c_i for $i \neq 1$ are zero and $c_1 \equiv c = 1$. The adjacency matrices are given by

$$A_0 = 1, \quad A_i = S^i + S^{-i} \quad \text{for} \quad i = 1, ..., k-1, \quad \text{and} \quad A_k = S^k, \tag{5-81}$$

where, S is the shift operator with period N, i.e., $S^N = I$. The idempotents are easily written as

$$E_0 = \frac{1}{N}J, \quad E_i = |\tilde{i}\rangle\langle\tilde{i}| + |\tilde{-i}\rangle\langle\tilde{-i}|, \quad \text{for} \quad i = 1, ..., k-1, \quad E_k = |\tilde{k}\rangle\langle\tilde{k}|, \tag{5-82}$$

where

$$|\tilde{i}\rangle := \frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ \omega^i \\ \vdots \\ \omega^{i(N-1)} \end{pmatrix}. \tag{5-83}$$

Therefore, we have $m_0 = m_k = 1$, $m_i = 2$, for $i = 1, ..., k-1$. Clearly, we have $|\Gamma_i(\alpha)| = 2$, for $i = 1, ..., k-1$ and $|\Gamma_k(\alpha)| = 1$. Also, from (5-81), it can be seen that the spectrum of A_i, is given by

$$P_{il} = 2\cos\frac{2\pi il}{N}, \quad l = 0, 1, ..., k \tag{5-84}$$

Now, by using (5-72), we obtain

$$R_{\alpha\beta^{(l)}} = \frac{1}{N}\sum_{i=1}^{k}\frac{m_i(1-\cos\frac{2\pi il}{N})}{1-\cos\frac{2\pi i}{N}} = \frac{1}{N}(2\sum_{i=1}^{k-1}\frac{1-\cos\frac{2\pi il}{N}}{1-\cos\frac{2\pi i}{N}} + \frac{1-(-1)^l}{2}). \tag{5-85}$$

For example, if $\beta \in \Gamma_1(\alpha)$ $(l = 1)$, we have

$$R_{\alpha\beta^{(1)}} = \frac{1}{N}(2(\frac{N}{2}-1)+1) = \frac{(N-1)}{N}. \tag{5-86}$$

In the limit of the large N, the cyclic network C_{2k} tend to the infinite line network which is the same as the root lattice A_1. In this case, the eigenvalues P_{il} tend to $\lambda = 2\cos lx$ and the

equation (5-85) is replaced with

$$R_{\alpha\beta^{(l)}} = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-\cos lx}{1-\cos x}dx. \tag{5-87}$$

The integral (5-87) can be evaluated by the method of residues and gives the following simple result

$$R_{\alpha\beta^{(l)}} = l, \quad \text{for all } \ \beta\in\Gamma_l(\alpha). \tag{5-88}$$

5.1.2 Hypercube network

For Hypercube network (known also as binary Hamming scheme denoted by $H(n,2)$) with $N=2^n$ vertices, the adjacency matrices are given by

$$A_i = \sum_{perm.}\underbrace{\sigma_x\otimes\sigma_x ...\otimes\sigma_x}_{i}\underbrace{\otimes I_2\otimes ...\otimes I_2}_{n-i}, \quad i=0,1,...,n, \tag{5-89}$$

where, the summation is taken over all possible nontrivial permutations. In fact, the underlying network is the cartesian product of n-tuples of complete network K_2. Also it can be shown that, the idempotents $\{E_0, E_1, ..., E_n\}$ are symmetric product of n-tuples of corresponding idempotents of complete network K_2. That is, we have

$$E_i = \sum_{perm.}\underbrace{E_-\otimes E_- ...\otimes E_-}_{i}\underbrace{\otimes E_+\otimes ...\otimes E_+}_{n-i}, \quad i=0,1,...,n, \tag{5-90}$$

where

$$E_{\pm} = \frac{1}{2}(I\pm\sigma_x). \tag{5-91}$$

It is well known that, the eigenvalues P_{il} are given by

$$P_{il} = K_l(i), \tag{5-92}$$

where $K_l(x)$ are the Krawtchouk polynomials defined as

$$K_l(x) = \sum_{i=1}^{l}\begin{pmatrix} x \\ i\end{pmatrix}\begin{pmatrix} n-x \\ l-i\end{pmatrix}(-1)^i. \tag{5-93}$$

Also, we have

$$\kappa_i = m_i = \frac{n!}{i!(n-i)!}. \tag{5-94}$$

Therefore, by using (5-72) we obtain effective resistance between two arbitrary nodes α and β, such that $\beta \in \Gamma_l(\alpha)$ as

$$R_{\alpha\beta^{(l)}} = \frac{2l!}{2^n n(n-1)...(n-l+1)} \sum_{k=1}^{n} \frac{n!(\frac{n(n-1)...(n-l+1)}{l!} - K_k(l))}{k!(n-k)! \sum_{i=1}^{n} c_i(\frac{n!}{i!(n-i)!} - K_k(i))} \tag{5-95}$$

For $n = 2$ (square) and $l = 1$, we have

$$R_{\alpha\beta^{(1)}} = \frac{1}{4} \sum_{i=1}^{2} \frac{2(2-K_i(1))}{i!(2-i)!(2(c_1+c_2) - c_1 K_i(1) - c_2 K_i(2))} = \frac{1}{4} \frac{3c_1+c_2}{c_1(c_1+c_2)}, \tag{5-96}$$

where, for $c_2 = 0$ and $c \equiv c_1 = 1$, we obtain the simple result

$$R_{\alpha\beta^{(1)}} = \frac{3}{4c} = \frac{3}{4}. \tag{5-97}$$

5.1.3 d-dimensional periodic networks

In this subsection we consider two examples of the networks which are underlying networks of association schemes constructed from finite root lattices of type A_2 and $A_1 \times A_1$ (for more details see [13]). These networks are known as hexagonal and square lattices, respectively. To this aim, first we briefly recall some of the main facts about the root lattices of type A_n.

a) Root lattices of type A_n

It is well known that a Coxeter-Dynkin diagram determines a system of simple roots in the Euclidean space E_n. The finite group W, generated by the reflections through the hyperplanes perpendicular to roots α_i, $i = 1, ..., n$

$$r_i(\beta) = \beta - 2\frac{(\alpha_i, \beta)}{(\alpha_i, \alpha_i)}\alpha_i \in R, \tag{5-98}$$

is called a Weyl group (for the theory of such groups, see [14] and [15]). An action of elements of the Weyl group W upon simple roots leads to a finite system of vectors, which is invariant with respect to W. A set of all these vectors is called a system of roots associated with a

given Coxeter-Dynkin diagram (for a description of the correspondence between simple Lie algebras and Coxeter-Dynkin diagrams, see, for example, [17]). It is proven that roots of R are linear combinations of simple roots with integral coefficients. Moreover, there exist no roots which are linear combinations of simple roots α_i, $i = 1, 2, ..., n$, both with positive and negative coefficients. The set of all linear combinations

$$Q = \{\sum_{i=1}^{n} a_i\alpha_i \ \mid \ a_i \in Z\} \equiv \bigoplus_i Z\alpha_i, \tag{5-99}$$

is called a root lattice corresponding to a given Coxeter-Dynkin diagram. Root system R which corresponds to Coxeter-Dynkin diagram of Lie algebra of the group $SU(n+1)$, gives root lattice A_n. For example root system A_2 (corresponding to lie algebra of SU(3)), where the roots form a regular hexagon and α and β are simple roots. This lattice is sometimes called hexagonal lattice or triangular lattice.

It is convenient to describe root lattice A_n and its Weyl group in the subspace of the Euclidean space E_{n+1}, given by the relation $x_1 + x_2 + ... + x_{n+1} = 0$, where x_1, x_2,..., x_{n+1} are the orthogonal coordinates of a point $x \in E_{n+1}$. The unit vectors in directions of these coordinates are denoted by e_j, respectively. Clearly, $e_i \perp e_j$, $i \neq j$. The set of roots is given by the vectors $\alpha_{ij} = e_i - e_j$ for $i \neq j$. The roots α_{ij}, with $i < j$ are positive and the roots $\alpha_i \equiv \alpha_{i,i+1} = e_i - e_{i+1}$ for $i = 1, ..., n$, constitute the system of simple roots.

Now, recall that the point group corresponding to a lattice is a group of geometric symmetries leaving a point of the lattice fixed. For any root lattice, the point group is the same as the group of all automorphisms of the root system (i.e., the group of all isomorphisms of the root system onto itself). Then, the Weyl group is a normal subgroup of this group of automorphisms [17]. It has been shown that [17], the point group of the root lattices is equal to the semidirect group $W \rtimes S_{C-D}$, where W is the corresponding Weyl group and S_{C-D} is the group of the symmetries of the Coxeter-Dynkin diagram of the lattice (all automorphisms which map the Coxeter-Dynkin diagram to itself). Then, one can see that the point group of the root lattice A_n is the semidirect product group $S_{n+1} \rtimes Z_2$, where S_{n+1} is the symmetric

group (for more details see [13, 17]).

Now, consider a d-dimensional lattice, periodic in each direction with period m and total number of $N = m^d$ vertices. Each vertex of the lattice corresponds to a basis state $|a\rangle$, where a is a d-component vector with components $a_1, ..., a_d \in \{0, 1, ..., m-1\}$. In the limit of the large m, this lattice tend to $\underbrace{A_1 \times ... \times A_1}_{d}$, the cartesian product of d root lattices A_1. In the following we show that, this lattice is an underlying network of association scheme which can be derived from finite abelian group $Z_m^{\otimes d} = \underbrace{Z_m \times ... \times Z_m}_{d}$ $(m \geq 3)$. Therefore, we can obtain effective resistances in terms of scheme's properties.

The generators of the lattice $\underbrace{A_1 \times ... \times A_1}_{d}$ are S_1,..., S_d where $S_i = I \otimes ... \otimes I \otimes \underbrace{S}_{i} \otimes I \otimes ... \otimes I$ with $S^m = I$. Then, one can show that the point group of the lattice is $Z_2^{\otimes d} \rtimes \mathbf{S}_d$, where $Z_2^{\otimes d}$ is the corresponding Wyle group generated by the reflections $S_1 \to S_1^{-1}; S_2 \to S_2^{-1}; ...; S_d \to S_d^{-1}$, and $\mathbf{S}_d$ is the symmetry group which contains all possible permutations of the simple roots $S_1, S_2, ..., S_d$ (recall that, this is the same as the symmetries of the corresponding Coxeter-Dynkin diagram of the lattice which consists of the d disconnected simple roots).

Then, the adjacency matrix of the underlying network (which is the same as the orbit $O(S_1)$) is

$$A = S_1 + ... + S_d + S_1^{-1} + ... + S_d^{-1}. \tag{5-100}$$

In fact, the orbits of the point group corresponding to the lattice form a partition $P = \{P_i, i = (i_1, ..., i_d)\}$ for $Z_m^{\otimes d}$. Then, the adjacency matrices A_i are defined as the sum of all elements of P_i in the regular representation, i.e., we define

$$A_i = \sum_{g \in P_i} g, \tag{5-101}$$

More clearly, one can see that

$$A_{\mathrm{i}=(i_1,i_2,...,i_d)} = O(S_1^{i_1} S_2^{i_2} ... S_d^{i_d}) = S_1^{i_1} S_2^{i_2} ... S_d^{i_d} + \text{perm.} + S_1^{-i_1} S_2^{i_2} ... S_d^{i_d} + \text{perm} + S_1^{i_1} S_2^{-i_2} S_3^{i_3} ... S_d^{i_d} + \text{perm.} +$$

$$... + S_1^{i_1} S_2^{i_2} ... S_{d-1}^{i_{d-1}} S_d^{-i_d} + \text{perm.} + S_1^{-i_1} S_2^{-i_2} S_3^{i_3} ... S_d^{i_d} + \text{perm.} + ... + S_1^{-i_1} S_2^{-i_2} ... S_d^{-i_d} + \text{perm.}. \tag{5-102}$$

where, the "perm." after each term, denotes all permutations of the indices $1, 2, ..., d$ in that term. From Eq.(5-102), it can be easily seen that for these networks, the spectrum of the adjacency matrices can be find easily, because the adjacency matrices are diagonalized by Fourier matrix $F_m \otimes ... \otimes F_m$, simultaneously. The corresponding idempotents are given by

$$E_{\mathbf{i}=(i_1,i_2,...,i_d)} = E_{i_1}\otimes E_{i_2}\otimes ...\otimes E_{i_d} + \text{perm.} + E_{-i_1}\otimes E_{i_2}\otimes ...\otimes E_{i_d} + \text{perm} + E_{i_1}\otimes E_{-i_2}\otimes E_{i_3}\otimes ...\otimes E_{i_d} + \text{perm.} +$$

$$... + E_{i_1}\otimes E_{i_2}\otimes ...\otimes E_{i_{d-1}}\otimes E_{-i_d} + \text{perm.} + E_{-i_1}\otimes E_{-i_2}\otimes E_{i_3}\otimes ...\otimes E_{i_d} + \text{perm.} + ... + E_{-i_1}\otimes E_{-i_2}\otimes ...\otimes E_{-i_d} + \text{perm.}, \tag{5-103}$$

where $E_i = |i\rangle\langle i|$ with $|i\rangle = \frac{1}{\sqrt{m}}(1, \omega, ..., \omega^{m-1})^t$ for $i = 0, 1, ..., m-1$ (see Eq.(5-82)). From (5-102), one can deduce that the eigenvalues of the adjacency matrices $A_{\mathbf{i}}$ are given by

$$\lambda^{(l_1,...,l_d)}_{\mathbf{i}=(i_1,i_2,...,i_d)} = 2[\cos\frac{2\pi(l_1i_1 + ... + l_di_d)}{m} + \text{perm.} + \cos\frac{2\pi(-l_1i_1 + l_2i_2 ... + l_di_d)}{m} + \text{perm.} +$$

$$... + \cos\frac{2\pi(l_1i_1 + ... + l_{d-1}i_{d-1} - l_di_d)}{m} + \text{perm.} + \cos\frac{2\pi(-l_1i_1 - l_2i_2 + l_3i_3 + ... + l_di_d)}{m} + \text{perm.} + ... +$$

$$\cos\frac{2\pi(-l_1i_1 - l_2i_2 - ... - l_ni_n + l_{n+1}i_{n+1} + ... + l_di_d)}{m} + \text{perm.}], \quad n := \lfloor d/2 \rfloor. \tag{5-104}$$

In the limit of the large size of the lattice, the eigenvalues tend to

$$\lambda^{(l_1,...,l_d)}(x_1, ..., x_d) = 2[\cos(l_1x_1 + ... + l_dx_d) + \text{perm.} + \cos(-l_1x_1 + l_2x_2 ... + l_dx_d) + \text{perm.} +$$

$$... + \cos(l_1x_1 + ... + l_{d-1}x_{d-1} - l_di_d) + \text{perm.} + \cos(-l_1x_1 - l_2x_2 + l_3x_3 + ... + l_dx_d) + \text{perm.} + ... +$$

$$\cos(-l_1x_1 - l_2x_2 - ... - l_nx_n + l_{n+1}x_{n+1} + ... + l_dx_d) + \text{perm.}], \tag{5-105}$$

where, $x_k = \lim_{i_k, m\to\infty} 2\pi i_k/m$ for $k = 1, 2, ..., d$. By assumption of $c_1 \equiv c = 1$ and $c_i = 0$ for all $i \neq 1$ and using the fact that $\upsilon_1 \equiv \upsilon = d$, the Eq.(5-72) implies that the effective resistances in the infinite d-dimensional lattice $A_1 \times ... \times A_1$ are obtained as

$$R_{\alpha\beta^{(l_1...l_d)}} = \frac{2}{\kappa_{\mathbf{l}}}\frac{1}{(2\pi)^d}\int_0^{2\pi} dx_1 ... \int_0^{2\pi} dx_d = \{\kappa_l - 2[\cos(l_1x_1 + ... + l_dx_d) + \text{perm.} +$$

$$\cos(-l_1x_1 + l_2x_2 ... + l_dx_d) + \text{perm.} + ... + \cos(l_1x_1 + ... + l_{d-1}x_{d-1} - l_di_d) + \text{perm.} +$$

$$\cos(-l_1x_1 - l_2x_2 + l_3x_3 + ... + l_dx_d) + \text{perm.} + ... + \cos(-l_1x_1 - l_2x_2 - ... - l_nx_n + l_{n+1}x_{n+1} + ... + l_dx_d) + \text{perm.}]\}$$

$$\{d - 2(\cos x_1 + \cos x_2 + ... + \cos x_d)\}^{-1}. \tag{5-106}$$

In the following we consider the special cases of two-dimensional ($d = 2$) periodic networks such that in the limit of the large size of the networks, they tend to the root lattices $A_1 \times A_1$ and A_2, respectively. The first case is called finite square network and the latter one is called finite hexagonal network (although, for $n > 3$, the underlying networks can be constructed similarly, but the networks do not possess so physical importance).

b) Finite square network

For this case ($d = 2$), the point group is the same as the Heisenberg group $H_2 \cong (Z_2 \times Z_2) \rtimes Z_2$. More clearly, we have $Z_2 \times Z_2 = \{e; S_1 \to S_1^{-1}, S_2 \to S_2; S_1 \to S_1, S_2 \to S_2^{-1}; S_1 \to S_1^{-1}, S_2 \to S_2^{-1}\}$ and the third cyclic group Z_2 is generated by the permutation $S_1 \leftrightarrow S_2$.

Now, we choose the ordering of elements of $Z_m \times Z_m$ as follows

$$V = \{e, a, ..., a^{m-1}, b, ab, ..., a^{m-1}b, ..., b^{m-1}, ab^{m-1}, ..., a^{m-1}b^{m-1}\}, \tag{5-107}$$

where $a^m = b^m = e$. We use the notation (k, l) for the element $a^k b^l$ of the group. Clearly, $(k, l)(k', l') = (k + k', l + l')$ and $(k, l)^{-1} = (-k, -l)$. Then the vertex set V of the network will be $\{(k, l) : k, l \in \{0, 1, ..., m - 1\}\}$. Then, the corresponding orbits are given by

$$P_{k_1 k_2} := O((k_1, k_2)), \tag{5-108}$$

where, $P_{00} = \{(0, 0)\}$ (in this case, the partition P is called homogeneous). In the regular representation of the group, for the corresponding adjacency matrices and the corresponding idempotents, we have

$$A_{\mathbf{k}=(k_1,k_2)} = \sum_{g \in O((k_1,k_2))} g = S_1^{k_1} S_2^{k_2} + S_1^{-k_1} S_2^{-k_2} + S_1^{k_2} S_2^{k_1} + S_1^{-k_2} S_2^{-k_1} + S_1^{k_2} S_2^{-k_1} + S_1^{-k_2} S_2^{k_1} +$$

$$S_1^{-k_1} S_2^{k_2} + S_1^{k_1} S_2^{-k_2}, \quad \text{for} \quad k_1 \neq k_2, \tag{5-109}$$

$$E_{\mathbf{k}=(k_1,k_2)} = E_{k_1} \otimes E_{k_2} + E_{-k_1} \otimes E_{-k_2} + E_{k_2} \otimes E_{k_1} + E_{-k_2} \otimes E_{-k_1} + E_{k_2} \otimes E_{-k_1} + E_{-k_2} \otimes E_{k_1} +$$

$$E_{-k_1} \otimes E_{k_2} + E_{k_1} \otimes E_{-k_2}, \quad \text{for} \quad k_1 \neq k_2 \tag{5-110}$$

respectively, and

$$A_{\mathbf{k}=(k_1,k_1)} = \sum_{g\in O((k_1,k_1))} g = S_1^{k_1}S_2^{k_1} + S_1^{-k_1}S_2^{-k_1} + S_1^{k_1}S_2^{-k_1} + S_1^{-k_1}S_2^{k_1}, \tag{5-111}$$

$$E_{\mathbf{k}=(k_1,k_1)} = E_{k_1}\otimes E_{k_1} + E_{-k_1}\otimes E_{-k_1} + E_{k_1}\otimes E_{-k_1} + E_{-k_1}\otimes E_{k_1}. \tag{5-112}$$

Therefore, the cardinalities of the associate classes $\Gamma_i(o)$ (κ_i), the valencies of the adjacency matrices and the ranks of the idempotents are given by

$$\kappa_{\mathbf{0}=(0,0)} = m_{\mathbf{0}=(0,0)} = 1, \quad \kappa_{\mathbf{k}=(k_1,k_2)} = m_{\mathbf{k}=(k_1,k_2)} = 8 \;\text{ for }\; 0\neq k_1\neq k_2\neq 0,$$

$$\text{and}\;\; \kappa_{\mathbf{k}=(k,k)} = m_{\mathbf{k}=(k,k)} = 4 \;\;,\;\; \kappa_{\mathbf{k}=(k,0)} = m_{\mathbf{k}=(k,0)} = 4. \tag{5-113}$$

The eigenvalues of the adjacency matrix $A_{\mathbf{k}=(k_1,k_2)}$ with $k_1\neq k_2$ are given by

$$\lambda_{ij}^{\mathbf{k}=(k_1,k_2)} = 2\{\cos\frac{2\pi(ik_1+jk_2)}{m} + \cos\frac{2\pi(ik_2+jk_1)}{m} + \cos\frac{2\pi(ik_2-jk_1)}{m} + \cos\frac{2\pi(ik_1-jk_2)}{m}\}, \tag{5-114}$$

where for $k_1 = k_2$, we have

$$\lambda_{ij}^{\mathbf{k}=(k_1,k_1)} = 2\{\cos\frac{2\pi(i+j)k_1}{m} + \cos\frac{2\pi(i-j)k_1}{m}\}. \tag{5-115}$$

Clearly, for finite square lattice we have $c_{(1,0)} \equiv c = 1$ and $c_{(i_1,i_2)} = 0$ for all $i_1\neq 1$ and $i_2\neq 0$. Then, by substituting (5-113) and (5-114) in (5-72), the effective resistances on the finite square lattice are given by

$$R_{\alpha\beta^{(l_1l_2)}} = \frac{1}{m^2\kappa_{(l_1l_2)}}\sum_{k_1,k_2}\frac{m_{(k_1k_2)}[\kappa_{(l_1l_2)} - 2(\cos\frac{2\pi(k_1l_1+k_2l_2)}{m} + \cos\frac{2\pi(k_1l_2+k_2l_1)}{m} + \cos\frac{2\pi(k_1l_2-k_2l_1)}{m} + \cos\frac{2\pi(k_1l_1-k_2l_2)}{m})]}{2-\cos 2\pi k_1/m - \cos 2\pi k_2/m}. \tag{5-116}$$

where, $R_{\alpha\beta^{(l_1l_2)}}$ denotes the effective resistances between α and all the nodes $\beta\in\Gamma_{\mathbf{l}=(l_1l_2)}(\alpha)$. For instance for $\beta\in\Gamma_{\mathbf{1}=(10)}(\alpha)$, we obtain

$$R_{\alpha\beta^{(10)}} = \frac{1}{2m^2}\sum_{k_1,k_2}\frac{m_{(k_1k_2)}[2-\cos(2\pi k_1)/m - \cos(2\pi k_2)/m]}{2-\cos(2\pi k_1)/m - \cos(2\pi k_2)/m} = \frac{R}{2m^2}\sum_{k_1,k_2} m_{(k_1k_2)} = \frac{1}{2m^2}.m^2 = \frac{R}{2}. \tag{5-117}$$

In the limit of the large size of the finite lattice, i.e., in the limit of $m \to \infty$, we have the infinite square lattice. In this limit the eigenvalues (5-114) tend to

$$\lambda_{x_1,x_2}^{\mathbf{l}=(l_1 l_2)} = 2[\cos(l_1x_1 + l_2x_2) + \cos(l_1x_2 + l_2x_1) + \cos(l_1x_2 - l_2x_1) + \cos(l_1x_1 - l_2x_2)]$$

where, $x_1 = \lim_{k_1,m\to\infty} 2\pi k_1/m$ and $x_2 = \lim_{k_2,m\to\infty} 2\pi k_2/m$. Then, the effective resistances are calculated as follows

$$R_{\alpha\beta^{(l_1 l_2)}} = \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{2-\cos(l_1x_1+l_2x_2)-\cos(l_1x_2+l_2x_1)-\cos(l_1x_2-l_2x_1)-\cos(l_1x_1-l_2x_2)}{2-\cos x_1-\cos x_2}dx_1dx_2. \tag{5-118}$$

c) Hexagonal network

Now, we consider the finite root lattice A_2 which is called hexagonal lattice. The point group of the lattice A_2 is $S_3 \rtimes Z_2$, where S_3 is the group of permutations of the simple roots together with the lowest root (all permutations of S_1, S_2 and $(S_1S_2)^{-1}$). With the same ordering of elements as before, the corresponding orbits are given by

$$P_{k_1k_2} := O((k_1, -k_2)), \tag{5-119}$$

where, $P_{00} = \{(0,0)\}$. Then, for the corresponding adjacency matrices and the corresponding idempotents, we have

$$A_{\mathbf{k}=(k_1,k_2)} = \sum_{g\in O((k_1,-k_2))} g = S_1^{k_1}S_2^{-k_2} + S_1^{-k_1}S_2^{k_2} + S_1^{-k_2}S_2^{k_1} + S_1^{k_2}S_2^{-k_1} + S_1^{-k_2}S_2^{k_1+k_2} + S_1^{k_2}S_2^{-(k_1+k_2)}+$$
$$S_1^{k_1+k_2}S_2^{k_2} + S_1^{-(k_1+k_2)}S_2^{-k_2} + S_1^{k_2}S_2^{k_1+k_2} + S_1^{-k_2}S_2^{-(k_1+k_2)} + S_1^{k_1+k_2}S_2^{k_1} + S_1^{-(k_1+k_2)}S_2^{-k_1}, \quad \text{for} \quad k_1 \neq k_2, \tag{5-120}$$

$$E_{\mathbf{k}=(k_1,k_2)} = E_{k_1}\otimes E_{-k_2} + E_{-k_1}\otimes E_{k_2} + E_{-k_2}\otimes E_{k_1} + E_{k_2}\otimes E_{-k_1} + E_{-k_2}\otimes E_{k_1+k_2} + E_{k_2}\otimes E_{-(k_1+k_2)}+$$
$$E_{k_1+k_2}\otimes E_{k_2} + E_{-(k_1+k_2)}\otimes E_{-k_2} + E_{k_2}\otimes E_{k_1+k_2} + E_{-k_2}\otimes E_{-(k_1+k_2)} + E_{k_1+k_2}\otimes E_{k_1} + E_{-(k_1+k_2)}\otimes E_{-k_1}, \quad \text{for} \quad k_1 \neq k_2 \tag{5-121}$$

respectively, and

$$A_{\mathbf{k}=(k,k)} = \sum_{g\in O((k,k))} g = S_1^kS_2^{-k} + S_1^{-k}S_2^{k} + S_1^kS_2^{-2k} + S_1^{-k}S_2^{2k} + S_1^{2k}S_2^{-k} + S_1^{-2k}S_2^{k}+$$

$$S_1^{2k}S_2^{k}+S_1^{-2k}S_2^{-k}+S_1^{k}S_2^{2k}+S_1^{-k}S_2^{-2k}, \tag{5-122}$$

$$E_{\mathbf{k}=(k,k)}=E_k\otimes E_{-k}+E_{-k}\otimes E_k+E_k\otimes E_{-2k}+E_{-k}\otimes E_{2k}+E_{2k}\otimes E_{-k}+E_{-2k}\otimes E_k+$$
$$E_{2k}\otimes E_k+E_{-2k}\otimes E_{-k}+E_k\otimes E_{2k}+E_{-k}\otimes E_{-2k}. \tag{5-123}$$

Therefore, the cardinalities of the associate classes $\Gamma_i(o)$ (a_i), the valencies of the adjacency matrices and the ranks of the idempotents are given by

$$\kappa_{\mathbf{0}=(0,0)}=m_{\mathbf{0}=(0,0)}=1,\quad \kappa_{\mathbf{k}=(k_1,k_2)}=m_{\mathbf{k}=(k_1,k_2)}=12 \ \text{ for } \ 0\neq k_1\neq k_2\neq 0,$$
$$\text{and}\ \ \kappa_{\mathbf{k}=(k,k)}=m_{\mathbf{k}=(k,k)}=10\ \ ,\ \ \kappa_{\mathbf{k}=(k,0)}=m_{\mathbf{k}=(k,0)}=6. \tag{5-124}$$

The eigenvalues of the adjacency matrix $A_{\mathbf{k}=(k_1,k_2)}$ with $k_1\neq k_2$ are given by

$$\lambda_{ij}^{(\mathbf{k})}=2\{\cos\frac{2\pi(ik_1-jk_2)}{m}+\cos\frac{2\pi(ik_2-jk_1)}{m}+\cos\frac{2\pi(ik_2-j(k_1+k_2))}{m}+\cos\frac{2\pi(i(k_1+k_2)-jk_2)}{m}+$$
$$\cos\frac{2\pi(ik_2+j(k_1+k_2))}{m}+\cos\frac{2\pi(i(k_1+k_2)+jk_1)}{m}\}. \tag{5-125}$$

where for $k_1=k_2\equiv k$, we have

$$\lambda_{ij}^{\mathbf{k}=(k,k)}=2\{\cos\frac{2\pi(i-j)k}{m}+\cos\frac{2\pi(i-2j)k}{m}+\cos\frac{2\pi(2i-j)k}{m}+\cos\frac{2\pi(2i+j)k}{m}+\cos\frac{2\pi(i+2j)k}{m}\}. \tag{5-126}$$

Then, similar to the case of finite square lattice, one can calculate effective resistances $R_{\alpha\beta^{(l)}}$.

In the limit of $m\to\infty$, we have the infinite hexagonal lattice. In this limit the eigenvalues (5-114) tend to

$$\lambda_{x_1,x_2}^{\mathbf{l}=(l_1l_2)}=2[\cos(l_1x_1-l_2x_2)+\cos(l_1x_2-l_2x_1)+\cos(l_1x_2-l_2(x_1+x_2))+\cos(l_1(x_1+x_2)-l_2x_2)+$$
$$\cos(l_1x_2+l_2(x_1+x_2))+\cos(l_1(x_1+x_2)+l_2x_1)]$$

where, x_1 and x_2 are defined as before. Then, the effective resistances are calculated as follows

$$R_{\alpha\beta^{(l_1l_2)}}=\frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{3-\cos(l_1x_1-l_2x_2)-\cos(l_1x_2-l_2x_1)-\cos(l_1x_2-l_2(x_1+x_2))-}{3-\cos x_1-\cos x_2-\cos(x_1+x_2)}$$
$$\frac{\cos(l_1(x_1+x_2)-l_2x_2)-\cos(l_1x_2+l_2(x_1+x_2))-\cos(l_1(x_1+x_2)+l_2x_1)}{3-\cos x_1-\cos x_2-\cos(x_1+x_2)}dx_1dx_2. \tag{5-127}$$

5.1.4 An example of the underlying networks of group association schemes

In this subsection we consider underlying network of group association scheme S_n and .

Symmetric group S_n

The symmetric group S_n is ambivalent in the sense that its conjugacy classes are real, i.e., $C_i = C_i^{-1}$ for all i and so form a symmetric association scheme.

As it is well known, for the group S_n, conjugacy classes are determined by the cycle structures of elements when they are expressed in the usual cycle notation. The useful notation for describing the cycle structure is the cycle type $[\nu_1, \nu_2, ..., \nu_n]$, which is the listing of number of cycles of each length (i.e, ν_1 is the number of one cycles, ν_2 is that of two cycles and so on). Thus, the number of elements in a conjugacy class or stratum is given by

$$|C_{[\nu_1,\nu_2,...,\nu_n]}| = \frac{n!}{\nu_1!2^{\nu_2}\nu_2!...n^{\nu_n}\nu_n!}. \tag{5-128}$$

On the other hand a partition λ of n is a sequence $(\lambda_1, ..., \lambda_n)$ where $\lambda_1 \geq \cdots \geq \lambda_n$ and $\lambda_1+\cdots+\lambda_n = n$, where in terms of cycle types

$$\lambda_1 = \nu_1 + \nu_2 + \cdots + \nu_n, \quad \lambda_2 = \nu_2 + \nu_3 + \cdots + \nu_n, \quad \cdots, \quad \lambda_n = \nu_n. \tag{5-129}$$

The notation $\lambda \vdash n$ indicates that λ is a partition of n. There is one conjugacy class for each partition $\lambda \vdash n$ in S_n, which consists of those permutations having cycle structure described by λ. We denote by C_λ the conjugacy class of S_n consisting of all permutations having cycle structure described by λ. Therefore the number of conjugacy classes of S_n, namely diameter of its scheme is equal to the number of partitions of n, which grows approximately by $\frac{1}{4\pi\sqrt{3}}e^{\pi\sqrt{2n/3}}$.

We consider the case where the generating set consists of the set of all transposition, i.e, $C_1 = C_{[2,1,1,1,1...,1]}$. For the characters at the transposition, it is known that [18]

$$\chi_\lambda(\alpha_1) = \frac{2!(n-2)!dim(\rho_\lambda)}{n!}\sum_j \left(\begin{pmatrix} \lambda_j \\ 2 \end{pmatrix} - \begin{pmatrix} \lambda'_j \\ 2 \end{pmatrix}\right). \tag{5-130}$$

Here, λ' is the partition generated by transposing the Young diagram of λ, while λ'_j and λ_j are the j-th components of the partitions λ' and λ, and ρ_λ is the irreducible representation corresponding to partition λ.

Then the eigenvalues of the adjacency matrix can be written as

$$P_{\lambda 1}=\frac{d_\lambda k_1}{m_\lambda}\chi_\lambda(\alpha_1)=\sum_j\left(\begin{pmatrix}\lambda_j\\ 2\end{pmatrix}-\begin{pmatrix}\lambda'_j\\ 2\end{pmatrix}\right). \tag{5-131}$$

In the above calculation, we have used the following results for the characters of the n-cycles

$$\chi_\lambda\left((n)\right)=\begin{cases}(-1)^{n-k} & \text{for } \lambda=(k,1,\ldots,1),\ k\in\{1,\ldots,n\}\\ 0 & \text{otherwise}\end{cases}$$

and

$$\chi_{(k,1,\cdots,1)}\left(\mathrm{id}\right)=dim(\rho_{(k,1,\cdots,1)})=\begin{pmatrix}n-1\\ k-1\end{pmatrix},\quad P_{\lambda 1}=\frac{1}{2}(2nk-n^2-n).$$

Then, one can evaluate effective resistances by using the Eq.(5-80). In the following, we consider the underlying network of group association scheme S_4 with diameter $d=4$, in details. To do so, we use the conjugacy classes of S_4 given by Eq.(4-36) and the adjacency matrices $A_i=\bar{C}_i,\quad i=0,1,...,4$ which satisfy the following Bose-Mesner algebra

$$A^2=6A_0+3A_2+2A_3,\quad AA_2=4A+4A_4,\quad AA_3=A+2A_4,\quad AA_4=4A_2+4A_3,$$

$$A_2^2=8A_0+4A_2+8A_3,\quad A_2A_3=3A_2,\quad A_2A_4=4A+4A_4,\quad A_3^2=3A_0+2A_3,$$

$$A_3A_4=2A+A_4,\quad A_4^2=6A_0+3A_2. \tag{5-132}$$

By using the character table of the group S_4 and Eq. (5-79), one can obtain

$$P_{0k}=1,\ \ k=0,...,4,\quad P_{10}=P_{12}=P_{13}=-P_{11}=-P_{14}=6,\quad P_{20}=P_{23}=8,\quad P_{21}=P_{24}=0,$$

$$P_{22}=-4,\ P_{30}=3,\ P_{31}=-P_{33}=-P_{34}=1,\ P_{32}=0,\ P_{40}=6,\ P_{41}=P_{43}=-P_{44}=-2,\ P_{42}=0. \tag{5-133}$$

Now, by using the Eq.(5-72), we obtain

$$R_{\alpha\beta^{(1)}}=\frac{1}{6}\{\frac{1}{12c_1+2c_3+8c_4}+\frac{9}{12c_1+8c_2+4c_3+4c_4}\},$$
$$R_{\alpha\beta^{(2)}}=\frac{1}{36}\{\frac{3}{12c_1+2c_3+8c_4}+\frac{20}{12c_2+3c_3+6c_4}-\frac{9}{4c_3+8c_4}+\frac{27}{12c_1+8c_2+4c_3+4c_4}\},$$
$$R_{\alpha\beta^{(3)}}=\frac{1}{18}\{\frac{1}{12c_1+2c_3+8c_4}+\frac{6}{12c_2+3c_3+6c_4}+\frac{18}{4c_3+8c_4}+\frac{18}{12c_1+8c_2+4c_3+4c_4}+\},$$

$$R_{\alpha\beta^{(4)}}=\frac{1}{48}\{\frac{5}{12c_1+2c_3+8c_4}+\frac{16}{12c_2+3c_3+6c_4}+\frac{45}{4c_3+8c_4}+\frac{27}{12c_1+8c_2+4c_3+4c_4}\}, \quad (5\text{-}134)$$

where, $R_{\alpha\beta^{(i)}}$ denotes the effective resistance between the node α and all nodes $\beta\in\Gamma_i(\alpha)$ for $i=1,2,3,4$.

6 Conclusion

Based on stratification of underlying networks of association schemes and using their algebraic combinatoric structure such as Bose-Mesner algebra together with spectral techniques, evaluation of effective resistances on these networks was discussed. It was shown that, in these types of networks, the effective resistances between a node α and all nodes β belonging to the same stratum with respect to α are the same. Then, by assumption that all of the conductances except for one of them is zero, a procedure for evaluation of effective resistances on particular underlying networks for which all of adjacency matrices are written as polynomials of the first adjacency matrix A of the network, was given such that effective resistances can be evaluated without using the spectrum of the networks. Moreover, an explicit analytical formula for effective resistance between arbitrary nodes α,β of an underlying resistor network of an association schemes (where all of conductances are non-zero) was given in terms of the spectrum of the networks. In each case, evaluation of effective resistance on some important finite underlying networks of association schemes and their corresponding infinite networks was given.

Appendix A

In this appendix we show that, for underlying networks of association schemes with diameter d such that the adjacency matrix A (or any of the other adjacency matrices A_i, $i = 2, ..., d$ which gives a connected network) has $d+1$ distinct eigenvalues, all of the adjacency matrices are polynomials of A, i.e., $A_i = P_i(A)$, where P_i is not necessarily of degree i. To do so, let A be the adjacency matrix of the connected network with $d+1$ distinct eigenvalues P_{1k}, $k = 0, 1, ..., d$. Then, by using the structure of the Bose-Mesner algebra, i.e., Eq.(2-4), one can write

$$A^l = \sum_{k=0}^{d} (P_{1k})^l E_k,$$

or in the matrix form

$$\begin{pmatrix} \mathbf{1} & A & A^2 & \ldots & A^d \end{pmatrix}^t = V \begin{pmatrix} E_0 & E_1 & E_2 & \ldots & E_d \end{pmatrix}^t$$

where, V is the Vandermonde matrix

$$V = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ P_{10} & P_{11} & \ldots & P_{1d} \\ P_{10}^2 & P_{11}^2 & \ldots & P_{1d}^2 \\ \vdots & \vdots & \vdots & \vdots \\ P_{10}^d & P_{11}^d & \ldots & P_{1d}^d \end{pmatrix}.$$

Clearly V is invertible due to the distinctness of the eigenvalues P_{1k} for $k = 0, 1, ..., d$. Then, we have

$$\begin{pmatrix} E_0 & E_1 & E_2 & \ldots & E_d \end{pmatrix}^t = V^{-1} \begin{pmatrix} \mathbf{1} & A & A^2 & \ldots & A^d \end{pmatrix}^t.$$

Now, by using (2-4), we write the idempotents E_i in terms of A_i to obtain

$$\begin{pmatrix} E_0 & E_1 & E_2 & \ldots & E_d \end{pmatrix}^t = \frac{1}{n} Q \begin{pmatrix} \mathbf{1} & A & A_2 & \ldots & A_d \end{pmatrix}^t = V^{-1} \begin{pmatrix} \mathbf{1} & A & A^2 & \ldots & A^d \end{pmatrix}^t.$$

Therefore, the adjacency matrices A_i, $i = 0, 1, ..., d$ can be written as polynomials of A, i.e., we have

$$\begin{pmatrix} \mathbf{1} & A & A_2 & \ldots & A_d \end{pmatrix}^t = n(VQ)^{-1} \begin{pmatrix} \mathbf{1} & A & A^2 & \ldots & A^d \end{pmatrix}^t.$$

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